

Violation of the holographic bulk viscosity bound

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Abstract

Motivated by gauge theory/string theory correspondence, a lower bound on the bulk viscosity of strongly coupled gauge theory plasma was proposed in [1]. We consider strongly coupled $\mathcal{N} = 4$ supersymmetric Yang-Mills plasma compactified on a two-manifold of constant curvature β . We show that the effective $(1 + 1)$ -dimensional hydrodynamic description of the system is governed by the bulk viscosity violating the bound of [1], once $\beta < 0$.

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1 Introduction

Relativistic hydrodynamics is an effective description of a nearly-equilibrium system at time (length) scales that are much longer (larger) than any characteristic microscopic scale of a system. In the simplest case (with no conserved charges), the dynamics of the hydrodynamic fluctuations in the system on a $(p+1)$ -dimensional manifold \mathcal{M}_{p+1} with the metric

$$ds^2 = G_{\mu\nu} dx^\mu dx^\nu, \quad (1.1)$$

is governed by conservation of the stress-energy tensor $T_{\mu\nu}$,

$$\nabla_\nu T^{\mu\nu} = 0. \quad (1.2)$$

The stress-energy tensor includes both an equilibrium part (with local energy density \mathcal{E} and pressure P) and a dissipative part $\Pi^{\mu\nu}$,

$$T^{\mu\nu} = \mathcal{E} u^\mu u^\nu + P \Delta^{\mu\nu} + \Pi^{\mu\nu} \quad \text{where} \quad \Delta^{\mu\nu} = G^{\mu\nu} + u^\mu u^\nu. \quad (1.3)$$

Above, u^μ is the local $(p+1)$ -velocity of the fluid with $u^\mu u_\mu = -1$. Further, $\Pi^{\mu\nu} u_\nu = 0$. In phenomenological hydrodynamics, the dissipative term $\Pi^{\mu\nu}$,

$$\Pi^{\mu\nu} = \sum_{n=1}^{\infty} \Pi_n^{\mu\nu}, \quad (1.4)$$

can be represented as an infinite series expansion in velocity gradients (and curvatures, for a fluid in a curved background), with the coefficients of the expansion commonly referred to as transport coefficients. In (1.4) the subscript $_n$ denotes the total number of the velocity $\partial_{\alpha_1} \cdots \partial_{\alpha_i} u_\beta$ and/or the background metric $\partial_{\alpha_1} \cdots \partial_{\alpha_j} G_{\alpha\gamma}$ derivatives. The familiar example of the Navier-Stokes equations is obtained by truncating $\Pi^{\mu\nu}$ at linear order in this expansion

$$\Pi^{\mu\nu} = \Pi_1^{\mu\nu}(\eta, \zeta) = -\eta \sigma^{\mu\nu} - \zeta \Delta^{\mu\nu} \nabla \cdot u, \quad (1.5)$$

where (for $p > 1$)

$$\sigma^{\mu\nu} = 2\nabla^{(\mu} u^{\nu)} \equiv \Delta^{\mu\alpha} \Delta^{\nu\beta} (\nabla_\alpha u_\beta + \nabla_\beta u_\alpha) - \frac{2}{p} \Delta^{\mu\nu} (\Delta^{\alpha\beta} \nabla_\alpha u_\beta). \quad (1.6)$$

Notice that at this order in the hydrodynamic approximation we need to introduce only two transport coefficients, namely the shear¹ η and bulk ζ viscosities. At the second order in the derivative expansion, $\Pi_2^{\mu\nu}$, one needs to introduce five [2] or thirteen [3] additional transport coefficients (depending on whether or not the system is conformal).

Holographic gauge theory/string theory correspondence [4] provides a useful general guidance about hydrodynamic transport coefficients. Thus, in [5], partly motivated by this correspondence, the authors (KSS) proposed a bound on the ratio of the shear viscosity to the entropy ratio²

$$\frac{\eta}{s} \geq \frac{1}{4\pi}. \quad (1.7)$$

The KSS bound is obeyed in all known fluids in Nature. It is either saturated or satisfied in all explicit or phenomenological examples of holographic gauge theory plasma at infinite coupling [5–9]. While the bound survives the leading order 't Hooft coupling corrections in four-dimensional holographic conformal models [10–12], it can be violated in strongly coupled conformal gauge theories with fundamental matter³ [13–15]⁴.

In [1] we proposed a bound on the bulk viscosity⁵ of (infinitely) strongly coupled gauge theory plasma

$$\frac{\zeta}{\eta} \geq 2 \left(\frac{1}{p} - c_s^2 \right), \quad (1.8)$$

¹To define the shear viscosity one needs $p > 1$.

²We use $\hbar = k_B = 1$.

³More generally, in four-dimensional conformal theories with different central charges: $c \neq a$.

⁴See [16–20] for further discussion of the shear viscosity bound.

⁵See [21–23] for the bound (and its violation) of a particular second-order transport coefficient — “the effective relaxation time”.

or equivalently (using the universality of the holographic shear viscosity),

$$\frac{\zeta}{s} \geq \frac{1}{2\pi} \left(\frac{1}{p} - c_s^2 \right), \quad (1.9)$$

where c_s^2 is the speed of sound waves in plasma. In what follows we use the second version of the bound, eq. (1.9), as it is applicable even for $p = 1$. The bulk viscosity bound (1.9) is saturated in toroidal compactifications of conformal theories [1, 24]. It is satisfied in various examples of string theory embedding of the holographic gauge theory/gravity correspondence: the strongly coupled $\mathcal{N} = 2^*$ gauge theory plasma [25, 26], the cascading gauge theory plasma [27, 28], the mass-deformed $\mathcal{N} = 4$ SYM plasma with a non-vanishing $U(1)$ R-charge chemical potential [29]. The bulk viscosity bound (1.9) is further satisfied in some phenomenological models of the holographic correspondence [30, 31], however, it is violated in some other phenomenological model [32, 33].

In this paper we address the question whether the violation of the bulk viscosity bound (1.9) is limited to the phenomenological models of the holographic gauge/gravity correspondence. A natural first guess is to replicate the strategy in the study of the KSS shear viscosity bound (1.7), *i.e.*, to consider the effect of the gauge theory finite coupling and non-planar corrections on the purported bound. Unfortunately, such an approach can not lead to a reliable conclusion. Indeed, when the bulk viscosity originates from toroidal compactifications of conformal hydrodynamics (in which case the string theory dual corrections to finite coupling/non-planar corrections can be under control), the general arguments in [1] guarantee what the bulk viscosity bound continues to be saturated. On the other hand, a five-dimensional gravitational description of a generic non-conformal holographic correspondence involves scalar fields that originate from 3-form RR fluxes of type IIB supergravity — in this case the full set of the higher derivative corrections to the supergravity is unknown. Clearly, we need to focus on explicit string theory examples of the holographic correspondence in the supergravity approximation.

Our strategy in exploring the bound (1.9) is to consider compactifications of the strongly coupled conformal hydrodynamics on k -dimensional spatial manifolds \mathcal{M}_k , with $k \leq (p - 1)$, of constant curvature $\beta \neq 0$. Specifically, we consider hydrodynamic transport of strongly coupled $\mathcal{N} = 4$ $SU(N)$ SYM plasma on \mathcal{M}_2 ,

$$\mathcal{M}_2 = \begin{cases} S^2, & \implies \beta = +\frac{2}{L^2} \\ \Sigma^2 = H^2/G, & \implies \beta = -\frac{2}{L^2} \end{cases}, \quad (1.10)$$

where S^2 is a round two-sphere, and H^2 is a hyperbolic space and G is a discrete subgroup of its $SL(2, R)$ symmetry group. The quotient Σ^2 is assumed to be smooth and compact. L is a "radius" of \mathcal{M}_2 . Since we are left with a single infinitely large spatial direction, the low-energy effective description of the compactification is given by $(1+1)$ -dimensional hydrodynamics. Notice that the latter hydrodynamic description *can not* be obtained from the compactification of the curved-space $(3+1)$ -dimensional hydrodynamics of $\mathcal{N} = 4$ plasma — the gradients of the background metric along the \mathcal{M}_2 directions are generically⁶ large in the hydrodynamic limit, which invalidates the gradient expansion (1.4). Running ahead, we find that to leading order in $\frac{\beta}{T^2} \ll 1$, the \mathcal{M}_2 -compactified hydrodynamics is affected by the *third-order* $\mathcal{N} = 4$ SYM hydrodynamics on $\mathcal{M}_4 = R^{1,1} \times \mathcal{M}_2$. To first-order in the velocity gradients, the effective $(1+1)$ -dimensional hydrodynamic description of the theory is determined by a single transport coefficient — the bulk viscosity ζ . We extract ζ from the dispersion relation of the sound waves in $\mathcal{N} = 4$ plasma on $R^{1,1} \times \mathcal{M}_2$ propagating along the single uncompactified spatial direction, which we denote as ' z ',

$$\mathfrak{w} = c_s^2 \mathfrak{q} - i \pi \frac{\zeta}{s} \mathfrak{q}^2 + \mathcal{O}(\mathfrak{q}^3), \quad (1.11)$$

where

$$\mathfrak{w} = \frac{\omega}{2\pi T}, \quad \mathfrak{q} = \frac{q_z}{2\pi T}, \quad (1.12)$$

and s is the entropy density. The speed of sound waves is determined from the equation of state as follows

$$c_s^2 = \left. \frac{\partial P_{zz}}{\partial \mathcal{E}} \right|_{\beta=\text{const}}, \quad (1.13)$$

where \mathcal{E} is the energy density, and $P_{zz} \equiv T_{zz}$ is the equilibrium pressure in the plasma in the z -direction.

The rest of the paper is organized as follows. In section 2 we discuss the regular black hole solution in AdS_5 with the asymptotic boundary metric $R^{1,1} \times \mathcal{M}_2$, dual to an equilibrium thermal state of $\mathcal{N} = 4$ SYM plasma compactified on \mathcal{M}_2 . We compute the one-point correlation function of the boundary stress-energy tensor and find the general expression for the speed of the sound waves (1.13). For general values of $\frac{\beta}{T^2}$, the background geometry (and its thermodynamics) is found numerically. We present analytical results for the thermodynamics to leading order in $\frac{\beta}{T^2} \ll 1$. In section 3 we

⁶A characteristic dimensionless parameter here is $\frac{\beta}{T^2}$, while a typical hydrodynamic parameter is $\frac{\omega}{T}$ or $\frac{|\vec{q}|}{T}$.

compute the dispersion relation of the sound channel quasinormal modes [34] in the black hole geometry of section 2, and extract the speed of the sound waves c_s^2 as well as the sound waves attenuation coefficient $\Gamma = \pi \frac{\zeta}{s}$. We present analytic results to leading order at high temperatures, $\frac{\beta}{T^2} \ll 1$, and numerical results for generic values of $\frac{\beta}{T^2}$. In section 4 we discuss the bulk viscosity bound and its violation in $\mathcal{N} = 4$ SYM plasma on $R^{1,1} \times \mathcal{M}_2$. We conclude in section 5.

2 Thermodynamics of $\mathcal{N} = 4$ SYM plasma on $R^{1,1} \times \mathcal{M}_2$

Effective five-dimensional gravitational action describing $\mathcal{N} = 4$ $SU(N)$ SYM on $\mathcal{M}_4 = R^{1,1} \times \mathcal{M}_2$ in the planar limit ($g_{YM} \rightarrow 0$, $N \rightarrow \infty$ with $\lambda \equiv g_{YM}^2 N = \text{const}$), and for an infinite 't Hooft coupling $\lambda \rightarrow \infty$, is given by

$$S_5 = \frac{1}{16\pi G_5} \int_{\mathcal{M}_5, \partial\mathcal{M}_5=\mathcal{M}_4} d^5\xi \sqrt{-g} (R + 12) , \quad (2.1)$$

where without loss of generality we normalized the radius of curvature of asymptotically AdS_5 geometry to 1. Such a normalization implies

$$G_5 = \frac{\pi}{2N^2} . \quad (2.2)$$

The background geometry dual to a state of the theory with translational invariance along the z -direction and the symmetry of \mathcal{M}_2 is given by

$$ds_5^2 = -c_1^2 dt^2 + c_2^2 \frac{2}{\beta} (d\mathcal{M}_2)^2 + c_3^2 dz^2 + c_4^2 dr^2 , \quad (2.3)$$

where $c_i = c_i(r)$ and $(d\mathcal{M}_2)^2$ is the standard metric on \mathcal{M}_2 . Given (2.3) we find the following second order equations of motion

$$0 = c_1'' - \frac{c_1(c_2')^2}{3c_2^2} + c_1' \left(\frac{4c_2'}{3c_2} - \frac{c_4'}{c_4} + \frac{2c_3'}{3c_3} \right) - \frac{2c_1c_3c_2'}{3c_2c_3} - 2c_1c_4^2 + \frac{\beta}{6} \frac{c_4^2c_1}{c_2^2} , \quad (2.4)$$

$$0 = c_2'' + \frac{2(c_2')^2}{3c_2} + c_1' \left(\frac{c_2'}{3c_1} - \frac{c_3c_2'}{3c_3c_1} \right) + c_2' \left(\frac{c_3'}{3c_3} - \frac{c_4'}{c_4} \right) - 2c_4^2c_2 - \frac{\beta}{3} \frac{c_4^2}{c_2} , \quad (2.5)$$

$$0 = c_3'' - \frac{(c_2')^2c_3}{3c_2^2} - \frac{2}{3}c_1' \left(\frac{c_2c_3'}{c_2c_1} - \frac{c_3'}{c_1} \right) + \left(\frac{4c_2'}{3c_2} - \frac{c_4'}{c_4} \right) c_3' - 2c_4^2c_3 + \frac{\beta}{6} \frac{c_3c_4^2}{c_2^2} , \quad (2.6)$$

as well as the first order constraint

$$0 = 2\frac{c_2'c_1'c_2}{c_1} + (c_2')^2 + \frac{c_1'c_3'c_2^2}{c_3c_1} - \frac{\beta}{2}c_4^2 - 6c_4^2c_2^2 + 2\frac{c_3'c_2'c_2}{c_3} . \quad (2.7)$$

We explicitly verified that (2.7) is consistent with (2.4)-(2.6).

2.1 Asymptotics of the background geometry dual to a thermal state of the theory

To describe a thermal state of strongly coupled $\mathcal{N} = 4$ SYM plasma on $R^{1,1} \times \mathcal{M}_2$ we find it convenient to use the radial coordinate

$$x \equiv 1 - \frac{c_1}{c_3}, \quad (2.8)$$

and further introduce⁷

$$c_2(x) = \frac{a(x)g(x)}{(2x - x^2)^{1/4}}, \quad c_3(x) = \frac{a(x)}{(2x - x^2)^{1/4}}. \quad (2.9)$$

Near the boundary, *i.e.*, as $x \rightarrow 0_+$, the asymptotics of $\{a, g\}$ are given by

$$\begin{aligned} a &= \mu \left(1 + \frac{\beta\sqrt{2}}{18\mu^2} x^{1/2} + \left(a_{2,0} + \frac{\beta^2}{324\mu^4} - \frac{\beta^2}{384\mu^4} \ln x \right) x + \mathcal{O}(x^{3/2}) \right), \\ g &= 1 - \frac{\beta\sqrt{2}}{8\mu^2} x^{1/2} + \left(\frac{127\beta^2}{10368\mu^4} - 2a_{2,0} + \frac{\beta^2}{192\mu^4} \ln x \right) x + \mathcal{O}(x^{3/2}). \end{aligned} \quad (2.10)$$

They are characterized by two independent parameters: $\{\mu, a_{2,0}\}$. As we will see later, μ is related to the temperature of the thermal state, and $a_{2,0}$ determines the one-point correlation function of the boundary stress-energy tensor at thermal equilibrium. Near the regular horizon, *i.e.*, as $y \equiv (1 - x) \rightarrow 0_+$, the asymptotics of $\{a, g\}$ are given by

$$\begin{aligned} a &= \mu \left(a_0^h + \frac{1}{4} a_0^h (32g_1^h g_0^h (a_0^h)^2 - 1) y^2 + \mathcal{O}(y^4) \right), \\ g &= g_0^h + \frac{\beta}{\mu^2} g_1^h y^2 + \mathcal{O}(y^4). \end{aligned} \quad (2.11)$$

They are characterized by three independent parameters: $\{a_0^h, g_0^h, g_1^h\}$. Notice that, given μ , there are precisely four parameters

$$\{a_{2,0}, a_0^h, g_0^h, g_1^h\},$$

which is necessary to uniquely specify a solution for the second-order ODEs for $\{a, g\}$.

Given (2.11) we can compute the Hawking temperature T and the Bekenstein-Hawking entropy density s of the black hole geometry (2.3)

$$\left(\frac{2\pi T}{\mu} \right)^2 = \frac{1}{8g_0^h g_1^h}, \quad \frac{s}{\mu^3} = \frac{(a_0^h)^3 g_0^h}{4G_5}. \quad (2.12)$$

⁷The two second-order equations of motion for $\{a, g\}$ can be obtained from (2.4)-(2.7). As they are not very illuminated, we do not present them here.

For general $\frac{\beta}{\mu^2}$, the equations of motion for $\{a, g\}$ are too complicated to be solved analytically — we solve them numerically. Analytical solution is possible though to leading order in the dimensionless parameter $\frac{\beta}{\mu^2}$. Using the high-temperature parametrization,

$$a(x) = \mu \left(1 + \frac{\beta}{\mu^2} a_1(x) + \mathcal{O}\left(\frac{\beta^2}{\mu^4}\right) \right), \quad g(x) = 1 + \frac{\beta}{\mu^2} g_1(x) + \mathcal{O}\left(\frac{\beta^2}{\mu^4}\right), \quad (2.13)$$

we find

$$a_1(x) = \frac{1 + 2x - x^2}{24(2x - x^2)} \left(\ln(1 - x) + \operatorname{arctanh}\sqrt{2x - x^2} \right) + \frac{2x - x^2 - 2\sqrt{2x - x^2}}{48(2x - x^2)}, \quad (2.14)$$

$$g_1(x) = -\frac{1}{8} \operatorname{arctanh}\sqrt{2x - x^2} - \frac{1}{8} \ln(1 - x).$$

Given (2.14), we find⁸

$$a_{2,0} = -\frac{1}{16} \frac{\beta}{\mu^2} + \mathcal{O}\left(\frac{\beta^2}{\mu^4}\right), \quad a_0^h = \mu \left(1 + \frac{\beta}{\mu^2} \left(\frac{1}{12} \ln 2 - \frac{1}{48} \right) + \mathcal{O}\left(\frac{\beta^2}{\mu^4}\right) \right),$$

$$g_0^h = 1 - \frac{1}{8} \ln 2 \frac{\beta}{\mu^2} + \mathcal{O}\left(\frac{\beta^2}{\mu^4}\right), \quad g_1^h = \frac{1}{32} + \frac{\ln 2 - 1}{256} \frac{\beta}{\mu^2} + \mathcal{O}\left(\frac{\beta^2}{\mu^4}\right). \quad (2.15)$$

which implies (see (2.12))

$$\frac{\pi T}{\mu} = 1 + \frac{1}{16} \frac{\beta}{\mu^2} + \mathcal{O}\left(\frac{\beta^2}{\mu^4}\right), \quad \frac{s}{\mu^3} = \frac{1}{4G_5} \left(1 - \frac{1}{16} \frac{\beta}{\mu^2} + \mathcal{O}\left(\frac{\beta^2}{\mu^4}\right) \right). \quad (2.16)$$

2.2 Holographic renormalization and the boundary stress-energy tensor

Holographic renormalization of strongly coupled $\mathcal{N} = 4$ SYM plasma has been extensively discussed in the literature, see [35] for example. To render (dual gravitational) correlation functions finite, the effective action (2.1) has to be supplemented with the following set of counterterms:

$$S_{ct} = -\frac{3}{8\pi G_5} \int_{\mathcal{M}_4, \epsilon=c_3^{-2}} \sqrt{-\gamma} \left(1 + \frac{1}{2} \hat{P} - \frac{1}{12} \left(\hat{P}^{kl} \hat{P}_{kl} - \hat{P}^2 \right) \ln \epsilon \right), \quad (2.17)$$

where γ is the metric (2.3) restricted to $c_3^{-2} = \epsilon$, and

$$\hat{P} = \gamma^{ij} \hat{P}_{ij}, \quad \hat{P}_{ij} = \frac{1}{2} \left(R_{ij} - \frac{1}{6} R \gamma_{ij} \right). \quad (2.18)$$

⁸We get $\mathcal{O}(\beta/\mu^2)$ coefficient of g_1^h we actually need to compute the second order correction, $g_2(x)$ in (2.13). This is a straightforward extension and we omit the details.

In practice, we use expressions for the regularized one-point correlation function of the stress-energy tensor obtained in [36], in particular, see eq. (3.52), in the conformal limit. We find:

$$\begin{aligned} 8\pi G_5 T_{tt} &= \frac{4\pi^2}{N^2} \mathcal{E} = \frac{3}{2} \mu^4 - 2\mu^4 a_{2,0} + \frac{1}{48} \beta^2 \ln \frac{\mu}{\Lambda}, \\ 8\pi G_5 T_{zz} &= \frac{4\pi^2}{N^2} P_{zz} = \frac{1}{2} \mu^4 + 2\mu^4 a_{2,0} - \frac{1}{48} \beta^2 \ln \frac{\mu}{\Lambda}, \end{aligned} \quad (2.19)$$

where Λ is an arbitrary (fixed) renormalization scale associated with the ambiguity⁹ of defining the stress-energy tensor of a theory on curved background manifold \mathcal{M}_4 with broken supersymmetry.

Given (2.19), it is straightforward to compute the speed of the sound waves (1.13) in \mathcal{M}_2 -compactified hydrodynamics of $\mathcal{N} = 4$ SYM plasma:

$$c_s^2 = \left. \frac{\partial P_{zz}}{\partial \mathcal{E}} \right|_{\beta=\text{const}} = \frac{1 + 4a_{2,0}(\hat{\beta}) - 2\hat{\beta} a'_{2,0}(\hat{\beta}) - \frac{1}{96}\hat{\beta}^2}{3 - 4a_{2,0}(\hat{\beta}) + 2\hat{\beta} a'_{2,0}(\hat{\beta}) + \frac{1}{96}\hat{\beta}^2}, \quad \hat{\beta} \equiv \frac{\beta}{\mu^2}, \quad (2.20)$$

where the prime denotes derivative with respect to $\hat{\beta}$. In section 3 we compare the thermodynamic prediction (2.20) for the speed of the sound waves with direct computations from the dispersion relation of the sound channel quasinormal modes.

Using the asymptotic expansions (2.15) and (2.16) we find

$$\begin{aligned} \mathcal{E} &= \frac{3\pi^2 N^2 T^4}{8} \left(1 - \frac{\beta}{6\pi^2 T^2} + \mathcal{O}\left(\frac{\beta^2}{T^4}\right) \right), \\ P_{zz} &= \frac{\pi^2 N^2 T^4}{8} \left(1 - \frac{\beta}{2\pi^2 T^2} + \mathcal{O}\left(\frac{\beta^2}{T^4}\right) \right), \end{aligned} \quad (2.21)$$

and¹⁰

$$c_s^2 = \frac{1}{3} - \frac{\beta}{18\mu^2} - \frac{\beta^2}{432\mu^4} + \mathcal{O}\left(\frac{\beta^3}{\mu^6}\right). \quad (2.22)$$

Notice from (2.16) that since

$$s = \frac{\pi^2 N^2 T^3}{2} \left(1 - \frac{\beta}{4\pi^2 T^2} + \mathcal{O}\left(\frac{\beta^2}{T^4}\right) \right), \quad (2.23)$$

the basic thermodynamic relations

$$-P_{zz} = \mathcal{F} = \mathcal{E} - Ts, \quad d\mathcal{E} = Tds, \quad (2.24)$$

are (analytically) satisfied to order $\mathcal{O}\left(\frac{\beta^2}{T^4}\right)$.

⁹The same ambiguity appears in computation of the thermal stress-energy tensor of strongly coupled $\mathcal{N} = 2^*$ plasma [37].

¹⁰To compute c_s^2 to order $\mathcal{O}(\beta^3/\mu^6)$ we do not need the $\mathcal{O}(\beta^2/\mu^4)$ coefficient in the expansion of $a_{2,0}$, see (2.15).

3 Sound channel quasinormal mode of AdS_5 black hole with $R^{1,1} \times \mathcal{M}_2$ asymptotic boundary

Following [34], the dispersion relation for the sound waves in \mathcal{M}_2 -compactified $\mathcal{N} = 4$ SYM plasma is identified with the dispersion relation for the sound channel quasinormal modes propagating in the z -direction of the black hole geometry (2.3). We briefly outline the construction of the corresponding fluctuations.

Consider the following decoupled set of metric fluctuations

$$\begin{aligned} g_{\mu\nu} &\rightarrow g_{\mu\nu} + h_{\mu\nu}, \\ h_{tt}(t, z, r) &= c_1^2(r) H_{tt}(r) e^{-i\omega t + i q_z z}, \\ h_{tz}(t, z, r) &= c_3^2(r) H_{tz}(r) e^{-i\omega t + i q_z z}, \\ h_{ij}(t, z, r) &= c_3^2(r) H_{ss}(r) e^{-i\omega t + i q_z z} \gamma_{ij}^{(\mathcal{M}_2)}, \\ h_{zz}(t, z, r) &= c_3^2(r) H_{zz}(r) e^{-i\omega t + i q_z z} \end{aligned} \quad (3.1)$$

where

$$\gamma_{ij}^{(\mathcal{M}_2)} d\xi^i d\xi^j = \frac{2}{\beta} (d\mathcal{M}_2)^2. \quad (3.2)$$

From the equations of motion for the fluctuations (3.1) we obtain 4 second-order linear ODEs for $\{H_{tt}, H_{tz}, H_{zz}, H_{ss}\}$ as well as 3 linear first-order constraints associated with the diffeomorphism-fixing conditions

$$h_{tr} = h_{zr} = h_{rr} = 0. \quad (3.3)$$

The combination

$$Z_H \equiv \frac{4\mathbf{q}}{\mathfrak{w}} H_{tz} + 2 H_{zz} - 2 H_{ss} \left(\frac{c_3 c_3'}{c_2 c_2'} - \frac{\mathbf{q}^2}{\mathfrak{w}^2} \frac{c_1 c_1'}{c_2 c_2'} \right) + 2 \frac{\mathbf{q}^2}{\mathfrak{w}^2} c_1^2 c_3^2 H_{tt}, \quad (3.4)$$

is invariant under the residual diffeomorphisms (for the gauge fixing (3.3)) and satisfies the following decoupled linear ODE¹¹

$$0 = Z_H'' + \mathcal{C}_1 Z_H' + \mathcal{C}_2 Z_H, \quad \mathcal{C}_i = \mathcal{C}_i \left[\{c_{1,2,3}(r)\}, \mathfrak{w}, \mathbf{q} \right]. \quad (3.5)$$

In terms of the radial coordinate (2.8), the hydrodynamic limit takes form

$$Z_H(x) = (1-x)^{-i\mathfrak{w}} \left(z_{H,0}(x) + i\mathbf{q} z_{H,1}(x) + \mathcal{O}(\mathbf{q}^2) \right), \quad \mathfrak{w} = c_s \mathbf{q} - i\Gamma \mathbf{q}^2 + \mathcal{O}(\mathbf{q}^3), \quad (3.6)$$

¹¹The expressions for \mathcal{C}_i are too long to be presented here — they are available from the author upon request.

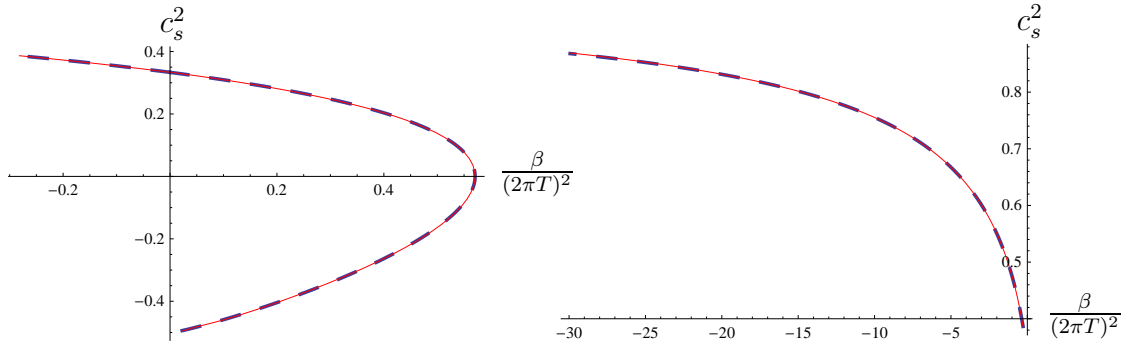


Figure 1: (Colour online) The speed of the sound waves in \mathcal{M}_2 -compactified $\mathcal{N} = 4$ SYM plasma as a function of $\frac{\beta}{(2\pi T)^2}$. The dashed blue curves indicate the speed of the sound obtained from the quasinormal mode dispersion relation, see (3.6). The solid red curves indicate the prediction for the speed of the sound waves from the equilibrium thermodynamics, see (2.20).

with the following boundary conditions

$$\begin{aligned} \lim_{x \rightarrow 1_-} z_{H,0} &= 1, & \lim_{x \rightarrow 1_-} z_{H,1} &= 0, \\ z_{H,i} &= \mathcal{O}(x), & \text{as } x &\rightarrow 0_+. \end{aligned} \quad (3.7)$$

We omit the details of solving the boundary value problem (3.5)-(3.7) for $\{z_{H,0}, z_{H,1}\}$ and present only the results¹². First of all, we find

$$\begin{aligned} \pm c_s &= \frac{1}{\sqrt{3}} - \frac{\sqrt{3}}{36} \frac{\beta}{\mu^2} - 0.004009(4) \frac{\beta^2}{\mu^4} + \mathcal{O}\left(\frac{\beta^3}{\mu^6}\right), \\ \Gamma &= \frac{1}{3} + \frac{\ln 2}{18} \frac{\beta}{\mu^2} + \mathcal{O}\left(\frac{\beta^2}{\mu^4}\right). \end{aligned} \quad (3.8)$$

Notice that the speed of the sound waves is in perfect agreement with the predictions from the thermodynamics (2.22) — the coefficient of $\frac{\beta^2}{\mu^4}$ agrees with an accuracy of $\propto 10^{-11}$.

Figure 1 presents the speed of the sound waves in \mathcal{M}_2 -compactified $\mathcal{N} = 4$ SYM plasma as a function of $\frac{\beta}{(2\pi T)^2}$. The dashed blue curves are obtained from the dispersion relation of the sound quasinormal modes (3.6), while the solid red curves indicate the prediction for the speed of the sound waves from the equilibrium thermodynamics, see

¹²For more details on the solution procedure see [26].

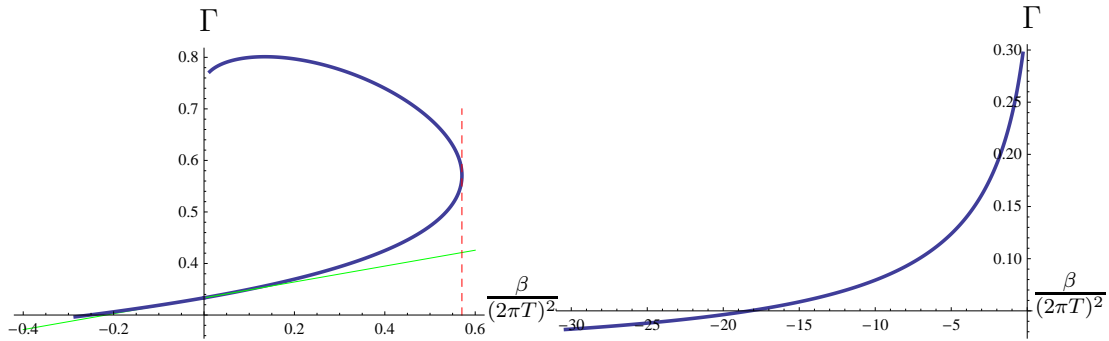


Figure 2: (Colour online) The attenuation of the sound waves (solid blue lines), $\Gamma = \pi_s^\zeta$, in \mathcal{M}_2 -compactified $\mathcal{N} = 4$ SYM plasma as a function of $\frac{\beta}{(2\pi T)^2}$. The solid green line indicates the high-temperature prediction (3.8). The vertical dashed red line indicates the critical temperature T_c at which the speed of the sound waves vanishes.

(2.20). Notice that at

$$\frac{\beta}{(2\pi T_c)^2} = 0.570580(8), \quad c_s^2 \Big|_{T=T_c} = 0, \quad (3.9)$$

the speed of the sound waves squared vanishes, and continuous to the unstable branch with $c_s^2 < 0$. In the vicinity of the critical point $c_s^2 \propto (1 - \frac{T_c}{T})^{1/2}$ — the same critical phenomena has been observed in $\mathcal{N} = 2^*$ gauge theory plasma [37], and the cascading gauge theory plasma [28].

Figure 2 presents the attenuation $\Gamma = \pi_s^\zeta$ of the sound waves in \mathcal{M}_2 -compactified $\mathcal{N} = 4$ SYM plasma as a function of $\frac{\beta}{(2\pi T)^2}$ (solid blue curves). The solid green line indicates the high-temperature prediction (3.8). The vertical dashed red line indicates the critical temperature T_c , see (3.9). Notice that as in the case of $\mathcal{N} = 2^*$ gauge theory plasma [1], the bulk viscosity remains finite at the critical point.

4 Bulk viscosity bound and its violation in $\mathcal{N} = 4$ SYM plasma on $R^{1,1} \times \mathcal{M}_2$

The bulk viscosity bound proposed in [1], in the case of strongly coupled $\mathcal{N} = 4$ SYM plasma compactified on \mathcal{M}_2 (see (1.10)) reads

$$\frac{\xi}{s} = \frac{\lambda}{2\pi} (1 - c_s^2), \quad \lambda \geq 1. \quad (4.1)$$

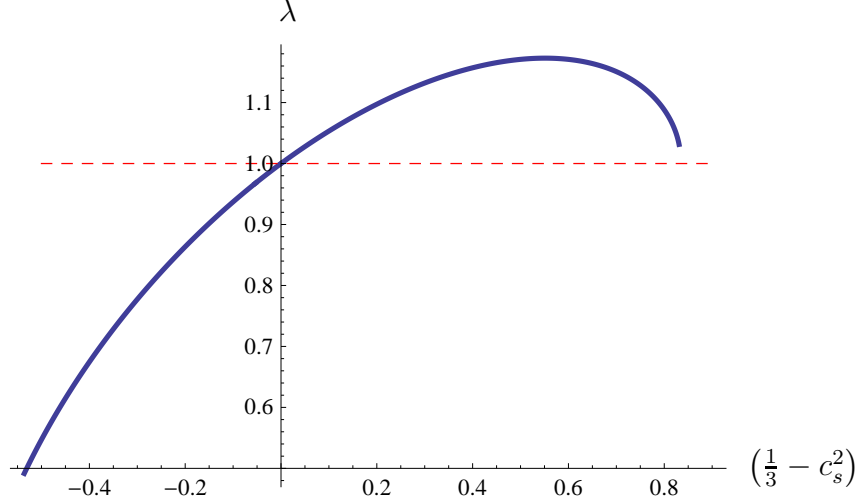


Figure 3: (Colour online) Violation of the bulk viscosity bound $\lambda \geq 1$ (see (4.1)) in $\mathcal{N} = 4$ SYM plasma compactified on \mathcal{M}_2 as a function of the conformal symmetry breaking parameter $(\frac{1}{3} - c_s^2)$.

From (3.8) we find

$$\lambda = 1 + \frac{2 \ln 2 - 1}{3} \frac{\beta}{(2\pi T)^2} + \mathcal{O}\left(\frac{\beta^2}{(2\pi T)^4}\right). \quad (4.2)$$

Clearly, if $\beta < 0$, *i.e.*, strongly coupled $\mathcal{N} = 4$ SYM plasma is compactified on Σ^2 , the bulk viscosity bound is violated. The leading (at high-temperature or equivalently small \mathcal{M}_2 curvature compactification) violation of the bulk viscosity bound is $\propto \beta \propto R$. It comes from 2-derivatives of the metric along \mathcal{M}_2 directions. When viewed from the perspective of higher-order hydrodynamics of $\mathcal{N} = 4$ SYM plasma on $\mathcal{M}_4 = R^{1,1} \times \mathcal{M}_2$, such a violation comes from the *third-order* dissipative term, $\Pi_3^{\mu\nu}$.

Figure 3 presents parameter λ in the bulk viscosity bound (4.1) as a function of the conformal symmetry breaking parameter $(\frac{1}{3} - c_s^2)$ of strongly coupled $\mathcal{N} = 4$ SYM plasma compactified on \mathcal{M}_2 . Notice that $\lambda < 1$ always, as long as $\beta < 0$, *i.e.*, the compactification manifold \mathcal{M}_2 is a constant curvature higher genus Riemann surface. The violation is rather strong at low temperatures: it is $\approx 51\%$ at the lowest temperature we accessed numerically

$$\min \left[\frac{\beta}{(2\pi T)^2} \right] = -30.379. \quad (4.3)$$

5 Conclusion

In this paper we constructed a specific string-theoretic counter-example to the bulk viscosity bound in (infinitely) strongly coupled gauge theory plasma proposed in [1]. We observed that compactification of the higher-dimensional hydrodynamics on curved manifolds results in Navier-Stokes (first-order) hydrodynamics with transport coefficients that are sensitive to higher-order dissipative terms of the higher-dimensional hydrodynamics. In the small-curvature limit of compactifications, the sensitivity starts with the third-order dissipative terms of the higher-dimensional conformal hydrodynamics. Since flat-space compactifications of the conformal hydrodynamics saturate the bulk viscosity bound of the effective lower-dimensional hydrodynamics [1], conformal hydrodynamics compactifications on curved manifolds are guaranteed to violate the bound for the judicious choice of the compactification manifold curvature. In a specific example of strongly coupled $\mathcal{N} = 4$ SYM plasma we showed that violation occurs for compactifications on negative curvature two-manifolds.

Since $\mathcal{N} = 4$ SYM contains conformally coupled scalars, one wonders whether the violation of the bulk viscosity bound can be attributed to the presence of tachyons¹³ in the theory, when compactified on Σ^2 . We do not believe this to be the case: in $\mathcal{N} = 2^*$ gauge theory plasma [37] one can study mass deformations with $\frac{m_b^2}{T^2} < 0$; such deformations do not violate the bound [1] since, for example at high-temperatures, the bulk viscosity is affected at order $\frac{m_b^4}{T^4}$.

We did not explore in details the issue of stability of \mathcal{M}_2 -compactifications of $\mathcal{N} = 4$ SYM discussed here. We did verify though that, at least at high temperature, there are no instabilities of metric fluctuations whose wave-functions are constant over \mathcal{M}_2 . Likewise, there are no instabilities of minimally coupled scalar (for example, a dilaton) in the background geometry (2.3), again, provided its wave-function is constant over \mathcal{M}_2 . In case of Σ^2 compactifications (which lead to the violation of the bulk viscosity bound) fluctuations with non-trivial wave-function on H^2 might be projected by the action of G on the quotient $\Sigma^2 = H^2/G$. The latter fact prevents making a general statement about the stability of the Σ^2 compactifications.

Finally, in this paper we considered compactifications of $\mathcal{N} = 4$ SYM on \mathcal{M}_2 which completely break the supersymmetry. It would be interesting to extend analysis presented here to supersymmetric (twisted) compactifications of $\mathcal{N} = 4$ SYM discussed

¹³Tachyons do not immediately imply the instabilities in the theory.

in [38].

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